An Algebraic Characterisation of First-Order Logic with Neighbour

Amaldev Manuel[1] Dhruv Nevatia[2]

[1]IIT Goa

[2]Chennai Mathematical Institute

First-order Logic with Neighbour

FO$(N, \mathrm{min}, \mathrm{max})$

$N(x, y)$:= positions x and y are adjacent

$$\varphi = P_a(\mathrm{min}) \wedge P_b(\mathrm{max}) \wedge \forall x \forall y\, N(x, y) \rightarrow \bigvee \begin{array}{l} P_a(x) \wedge P_b(y) \\ P_b(x) \wedge P_x(y) \end{array}$$

φ defines the language $(ab)^+$

First-order Logic with Neighbour –Contd.

$$N(x, y) := x + 1 = y \vee y + 1 = x$$

Hence $\mathrm{FO}(N, \min, \max) \subseteq \mathrm{FO}(+1) \subseteq \mathrm{FO}(<)$.

But not all $\mathrm{FO}(+1)$ languages are definable in $\mathrm{FO}(N, \min, \max)$.

Ehrenfeucht-Fraïsse Games

$$c \cdots c\, ab\, c \cdots c$$

$$\underbrace{c \cdots c}_{\gg k}\, ba\, \underbrace{c \cdots c}_{\gg k}$$

No FO(N, min, max) formula of quantifier depth k can distinguish between them.

Hence, $c^* abc^*$ is not definable in FO(N, min, max).

More generally, for any words x, w the following are not distinguishable.

$$w \cdots w\, x\, w^r \cdots w^r$$

$$\underbrace{w \cdots w}_{\gg k}\, x^r\, \underbrace{w^r \cdots w^r}_{\gg k}$$

Algebraic Characterisation

A language is definable in FO$(+1)$ iff it is recognised by a semidirect product $S * T$ where $S \in \mathrm{Acom}$ and $T \in \mathrm{D}$.

How can we characterise FO$(N, \min, \max)$?

Recognisability by Involution Semigroups

Definition (Involution Semigroup)

An *involution semigroup* $(S, \cdot, \star)$ is a semigroup $(S, \cdot)$ with a unary operation $\star : S \to S$ such that the following holds.

1. $(a^\star)^\star = a$
2. $(a \cdot b)^\star = b^\star \cdot a^\star$

Example

A^+ with concatenation and reverse.

Let $(S, \cdot, \star)$ and $(T, +, \dagger)$ be involution semigroups.

Definition

A morphism of involution semigroups, also called an involutory morphism, from S to T is a semigroup morphism $h\colon S \to T$ such that $h(a^\star) = h(a)^\dagger$, for all $a \in S$.

Definition (Recognition by Involution Semigroup)

$L \subseteq A^+$ is *recognised* by an involution semigroup $(S, \cdot, \star)$ if there is an involutory morphism $h : A^+ \to S$ and a subset $P \subseteq S$ such that $L = h^{-1}(P)$.

Bilateral Semidirect Product

Assume $(T, \cdot)$ has a two-sided action on $(S, +)$.

Definition (Bilateral Semidirect Product)

$S ** T$ is the semigroup on the set $S \times T$ with the product

$$(s_1, t_1)(s_2, t_2) = (s_1 t_2 + t_1 s_2, t_1 \cdot t_2)$$

Involutory Semidirect Product

Assume $(T, \cdot, \dagger)$ has a two-sided action on $(S, +, \star)$.

The action is *involutory* if $(tst')^\star = t'^\dagger s^\star t^\dagger$.

In such case $S ** T$ is an involution semigroup.

The action is *locally hermitian* if $exe^\dagger = ex^\star e^\dagger$.

In such case the product is called *locally hermitian semidirect product.*

An Algebraic Characterisation

Theorem

A language is definable in $\mathrm{FO}(N, \min, \max)$ *if and only if it is recognised by a locally hermitian semidirect product of an aperiodic commutative involution semigroup and a locally trivial involution semigroup.*

- The above theorem characterises the logic, though not a decidable one.
- It remains to eliminate the semidirect product.

Thank You