How do we compare transducers?

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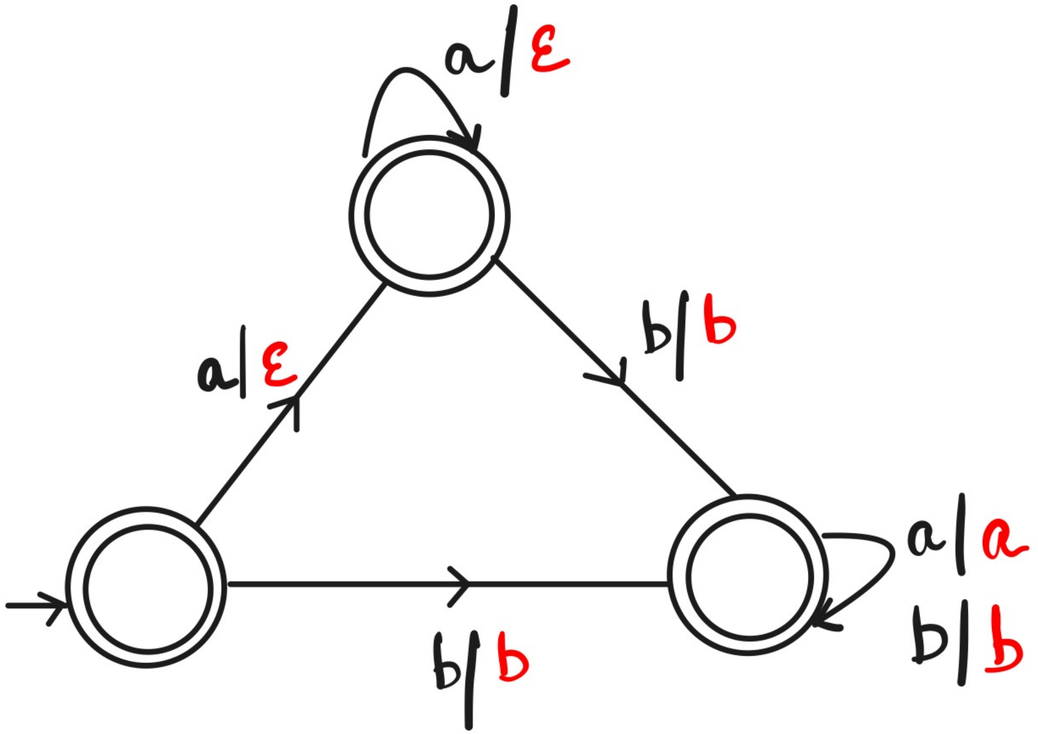
Joint work with **C. Aiswarya** (CMI) and **Saina Sunny** (IIT Goa)

Based on: 1. Edit Distance of Finite State Transducers (ICALP 2024)

2. Deciding Conjugacy of Rational Relation (DLT 2024)

Transducers

- Finite state machines that produce output.



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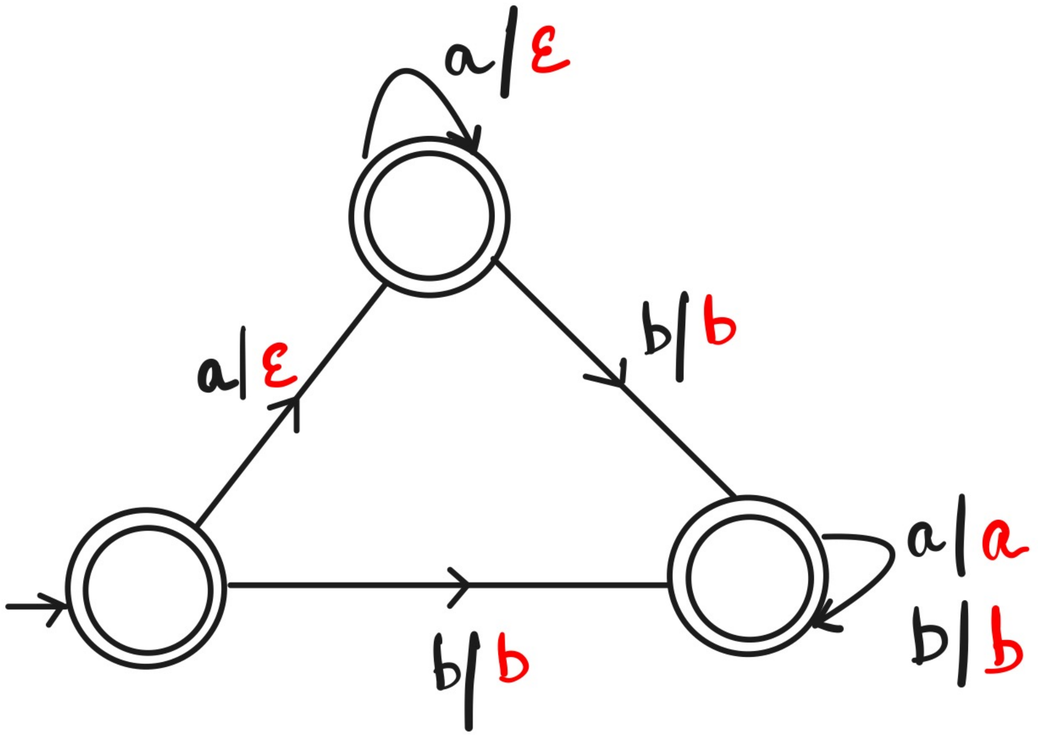

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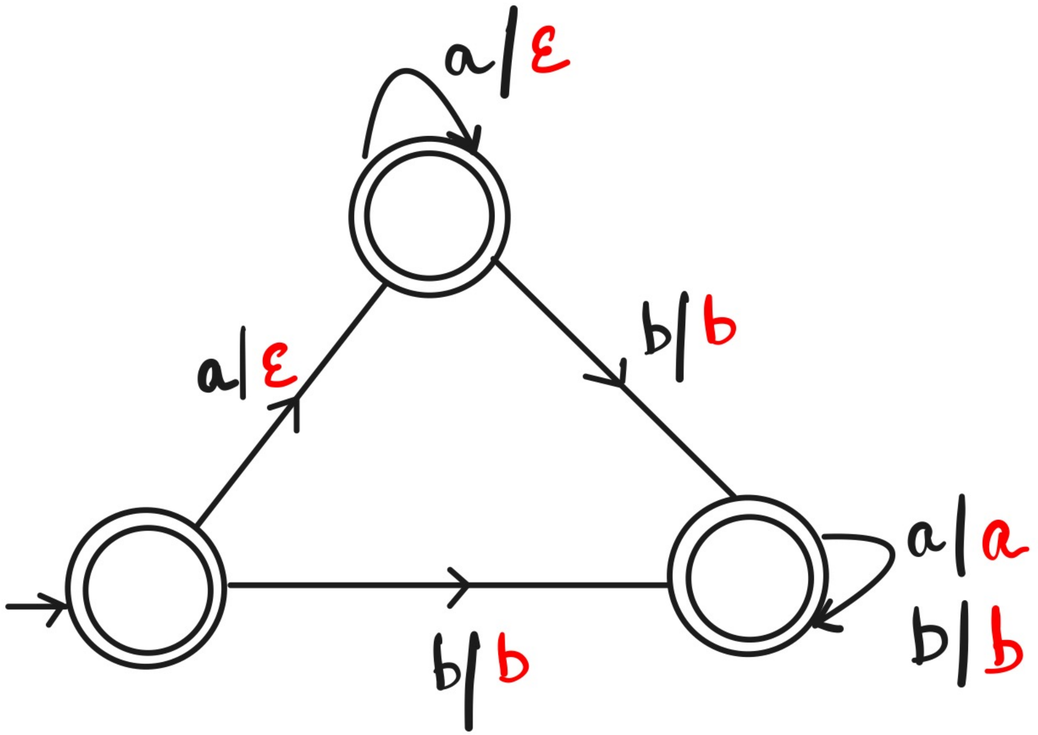

Transducers

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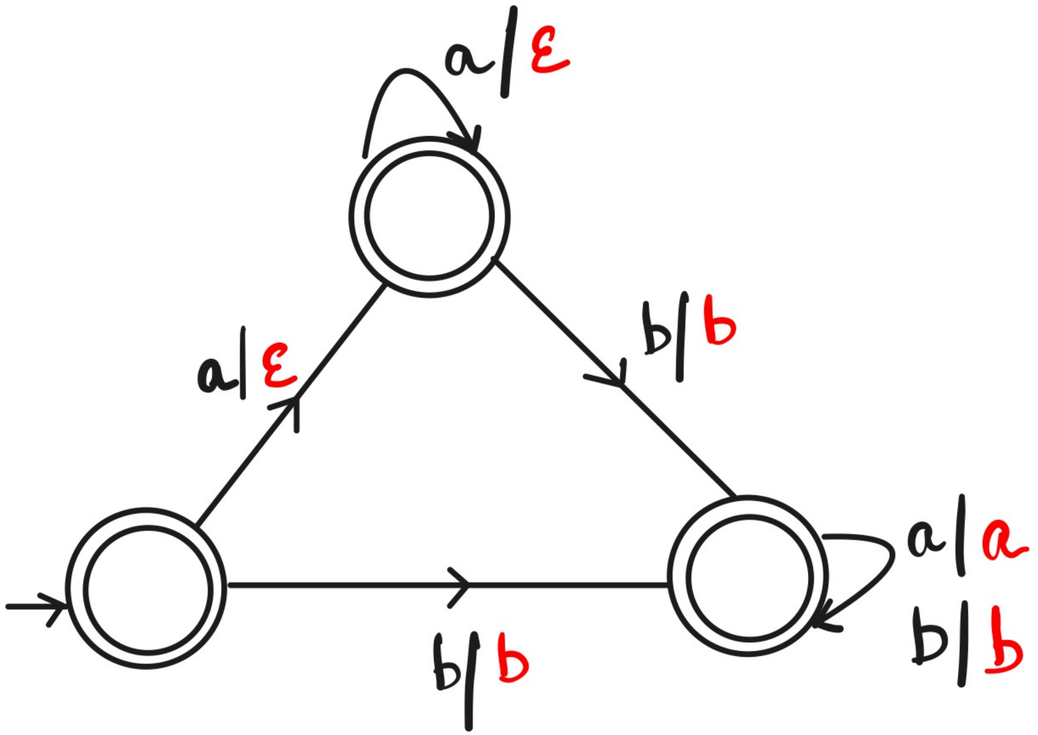
  - Sequential (DFA)
  - Rational (Unambiguous NFA)



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- The number of such edits needed can indicate some “distance” between two words

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- T_1 and T_2 are close if $d(T_1, T_2)$ is finite.

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- Levenshtein Edit Distance

$d(u,v) =$ Minimum number of insertions, deletions and substitutions required to convert u to v

Some metrics

- Discrete metric
- Hamming Distance
- Levenshtein Edit Distance - ins,del,sub
- Conjugacy distance - cyclic shifts
- Transposition distance - adjacent transpositions
- Damerau-Levenshtein Edit Distance - insertion, deletions, transpositions and letter-to-letter substitutions

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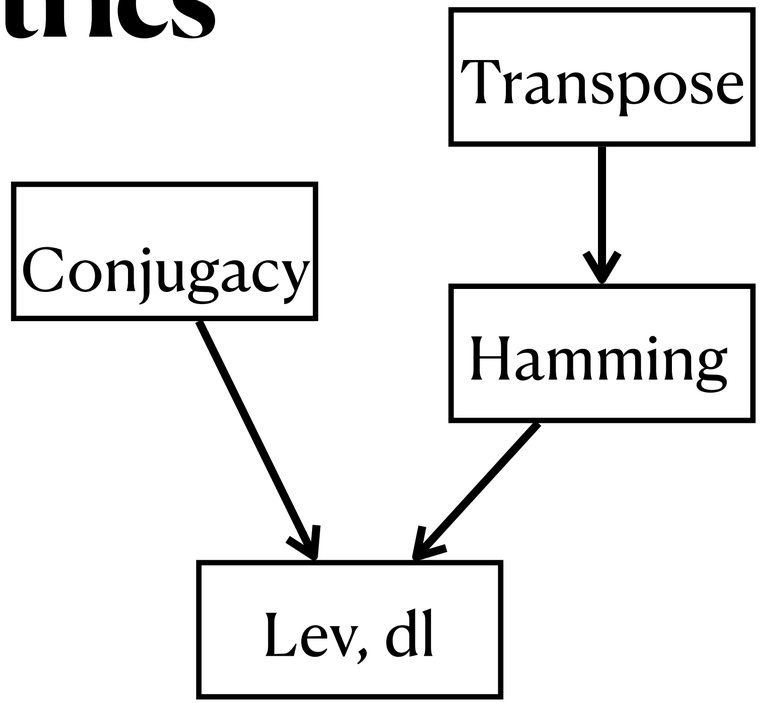

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(Distance)

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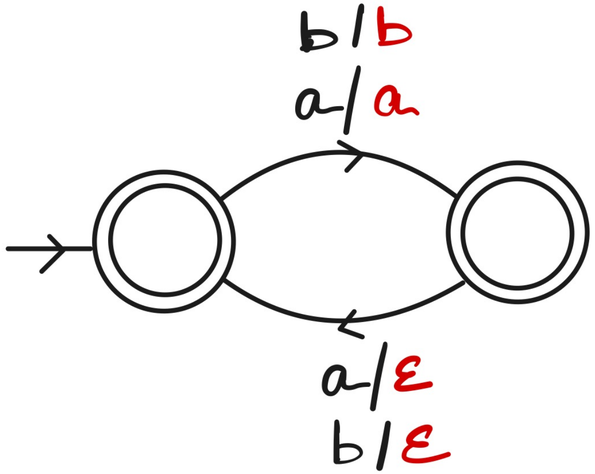
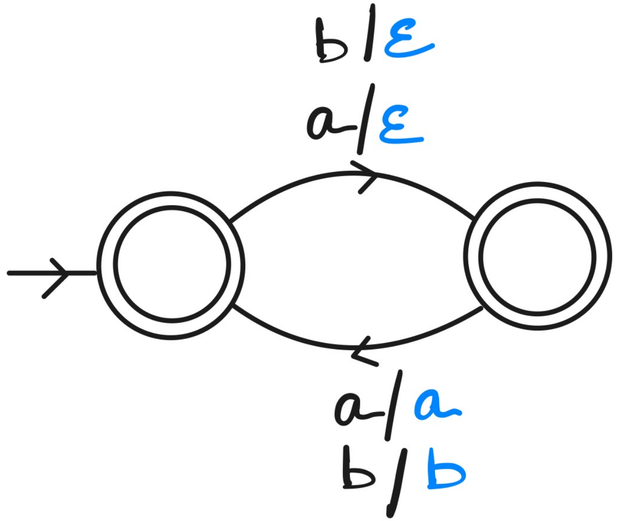
Unbounded Edit Distance - Example



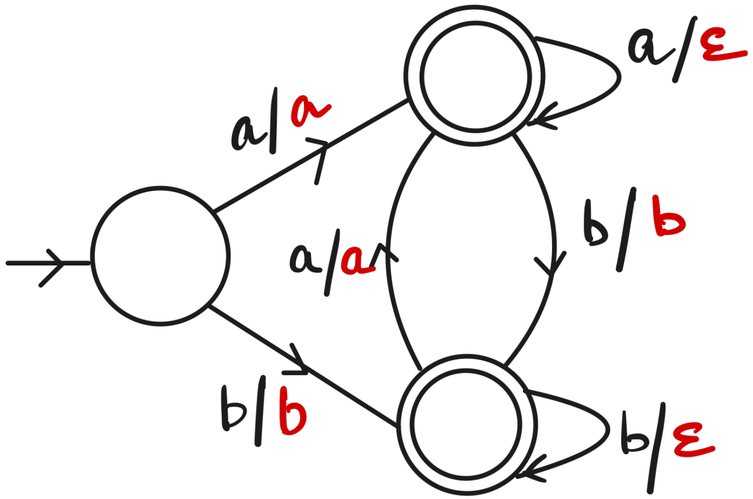


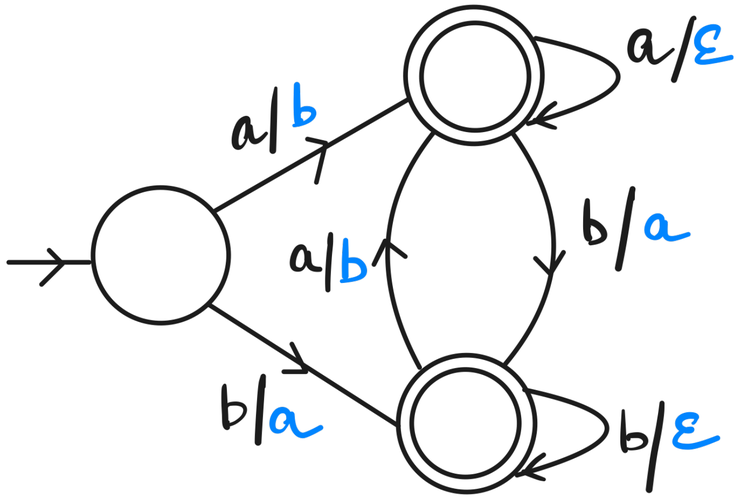
Output letters at odd positions

Output letters at even positions

$$(ab)^n \rightarrow (a^n, b^n)$$

Bounded Lev Edit Distance - Example





- For each block of a, output a
- For each block of b, output b

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$$aaabbbabbba \rightarrow (ababa, babab)$$

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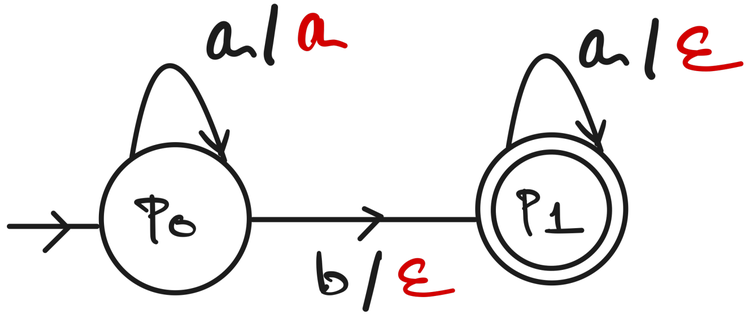

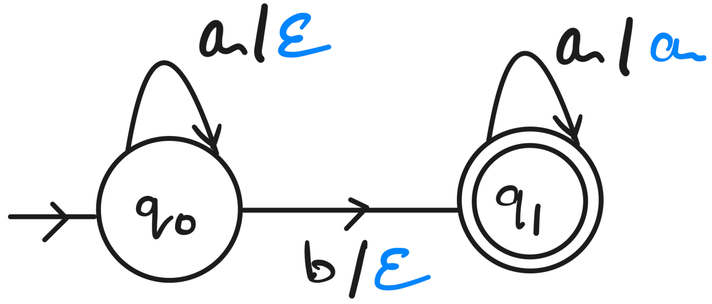

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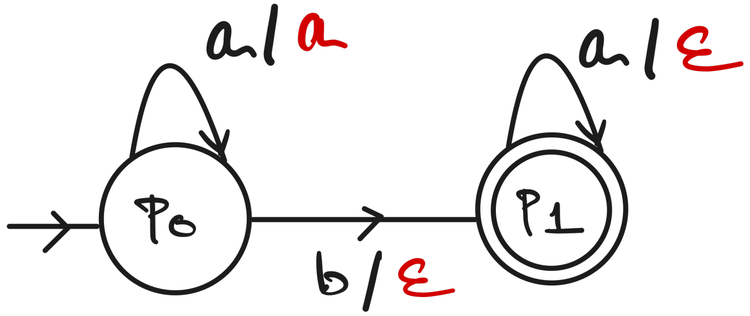

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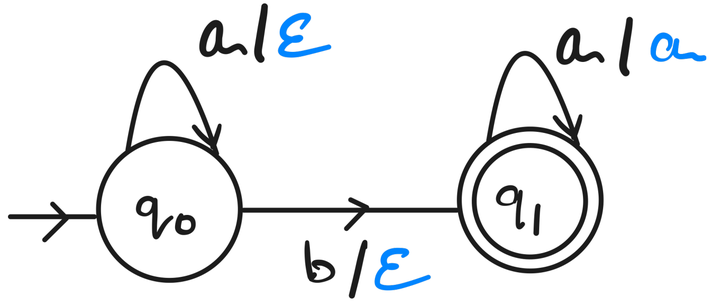

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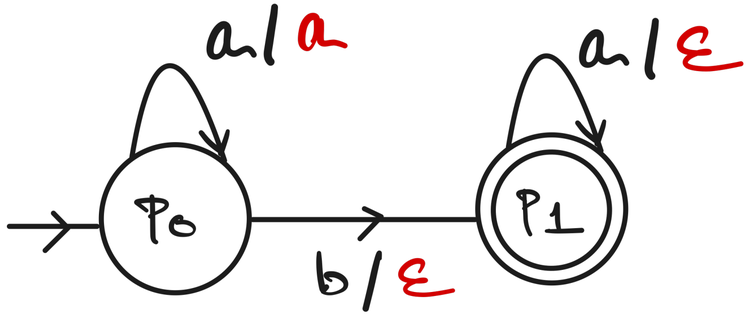

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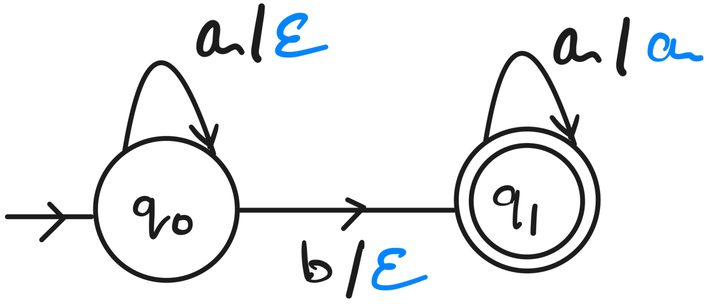

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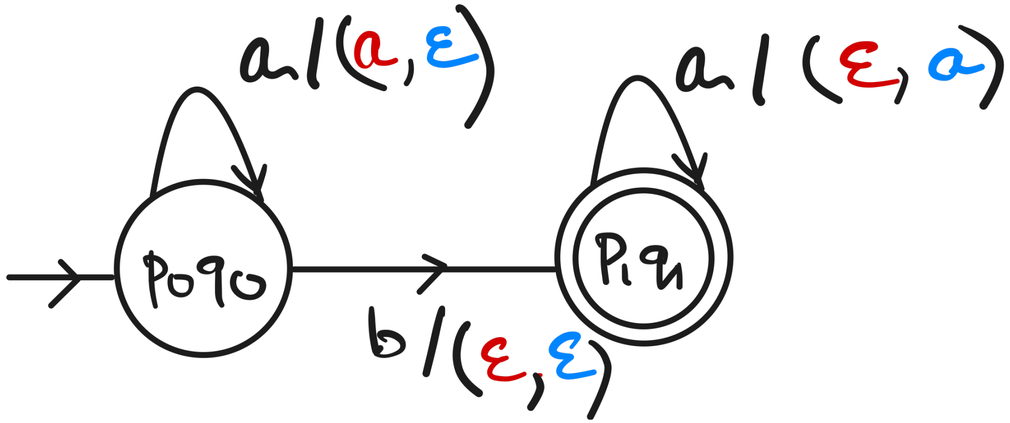

$$aaba \rightarrow (a, \epsilon) \cdot (a, \epsilon) \cdot (\epsilon, \epsilon) \cdot (\epsilon, a) = (aa, a)$$

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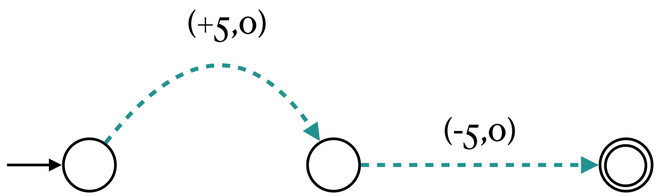

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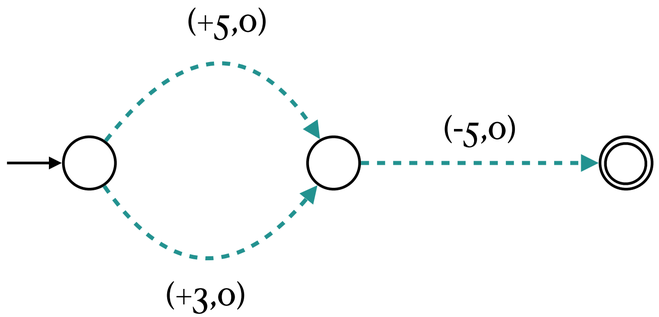

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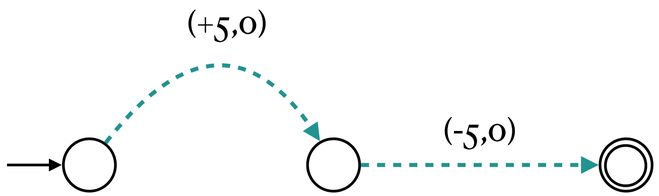

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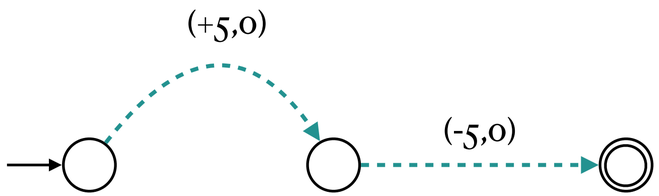

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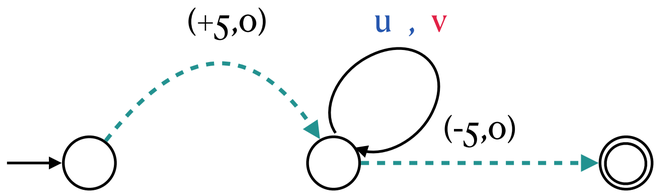

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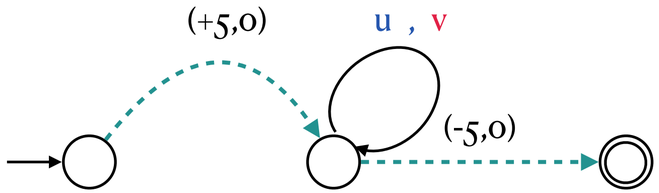

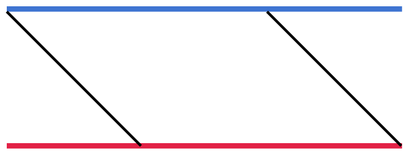

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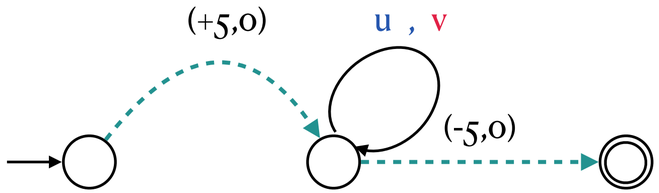

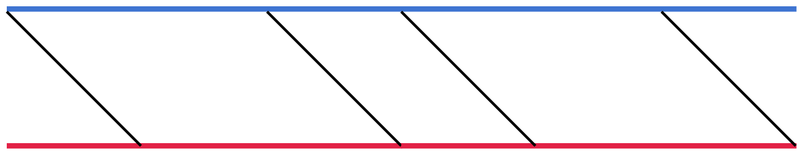

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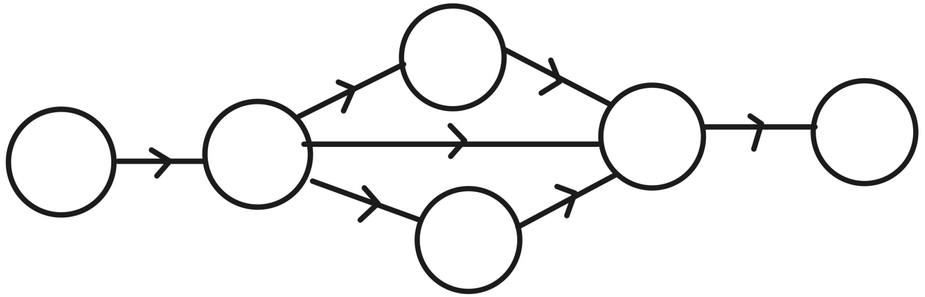
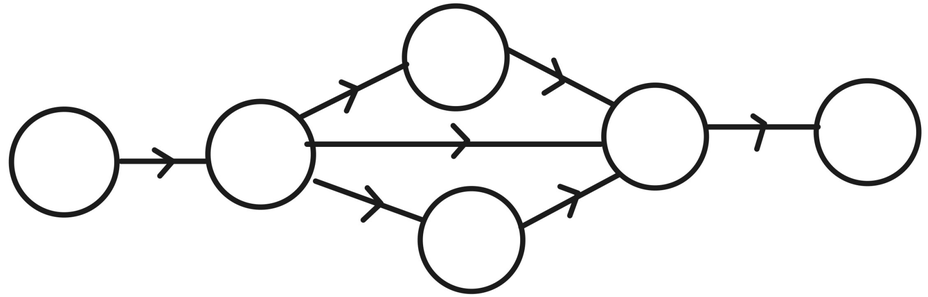
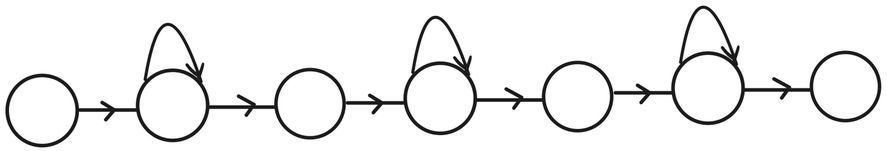


Closeness is decidable

For Levenshtein

- Consider the cartesian product
- If it is a DAG -> bounded
- If it has simple cycles





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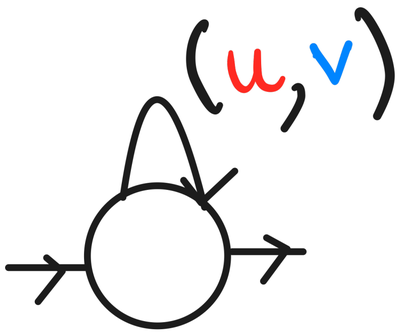

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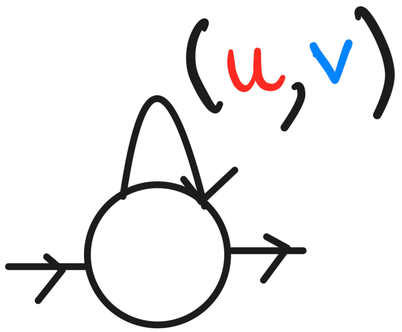

$$x\,y\,x\,y\,\cdots\,x\,y\,x\,y$$

$$y\,x\,y\,x\,\cdots\,y\,x\,y\,x$$

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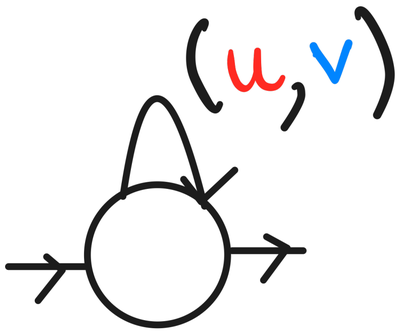

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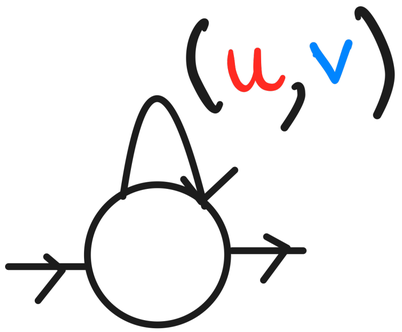
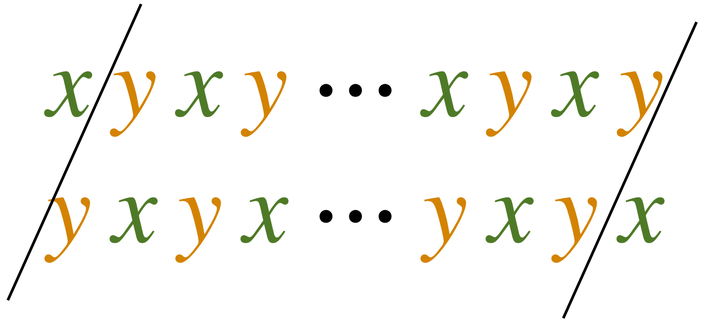
Conjugates

- u and v are conjugates if $u = xy$ and $v = yx$

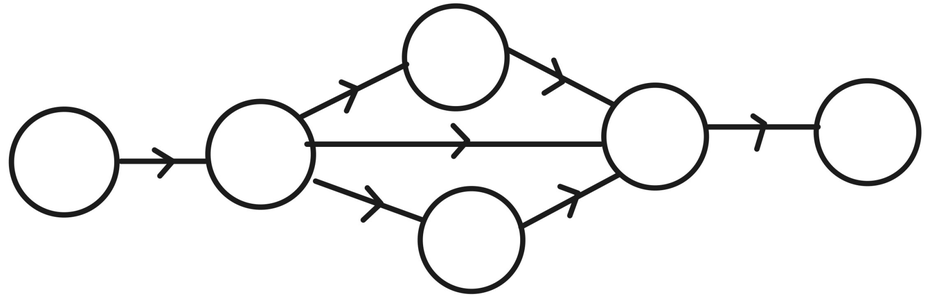


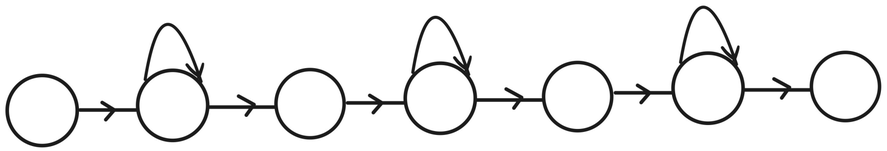


Closeness is decidable

For Levenstein

- Consider the cartesian product
- If it is a DAG -> bounded
- If it has simple cycles
- Check if each of them is conjugate
- If there are multiple cycle?





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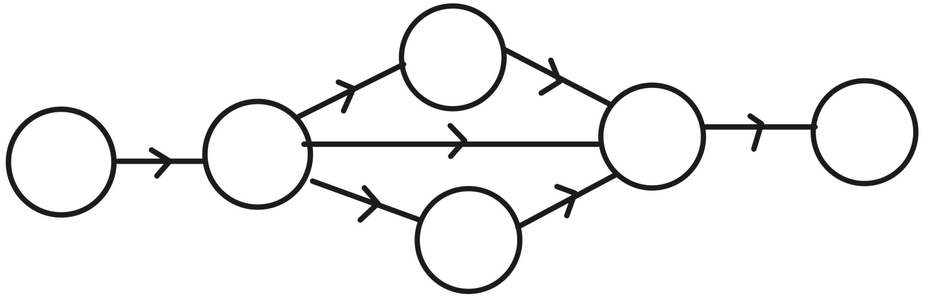
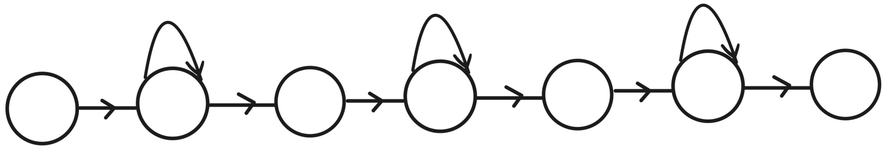
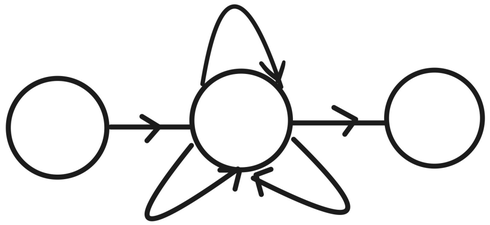
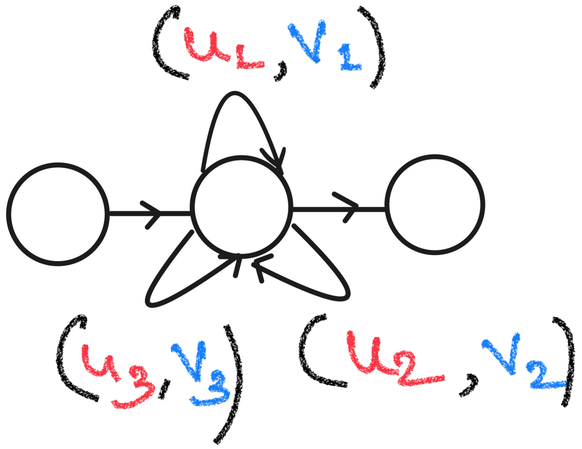




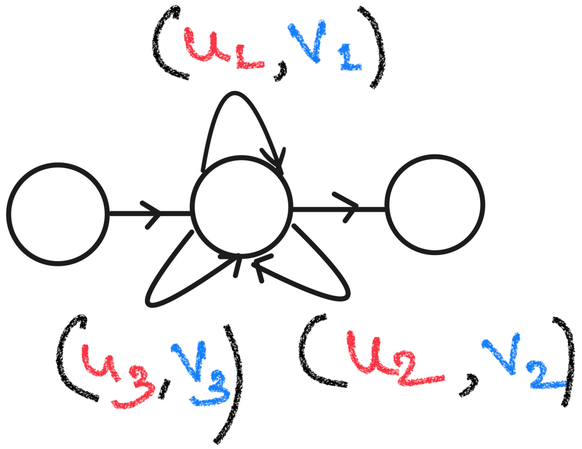


More on conjugates



where each u_i and v_i are conjugates

More on conjugates



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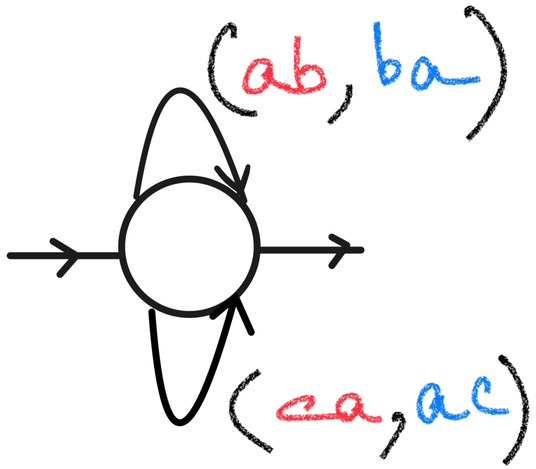

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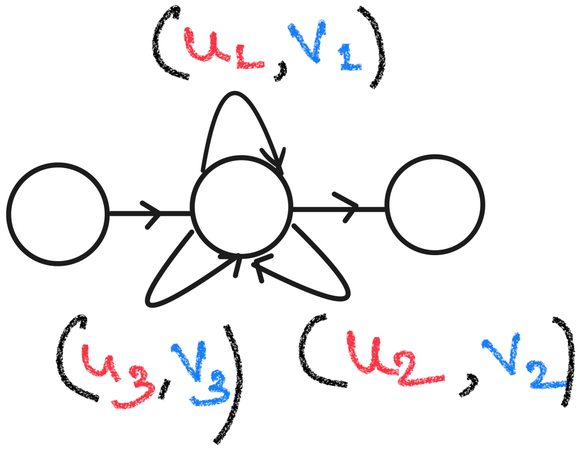

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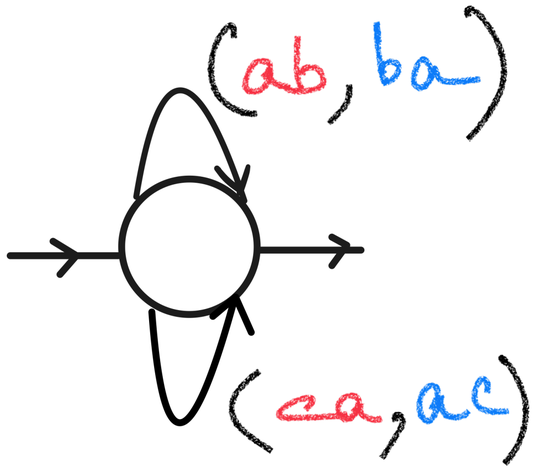
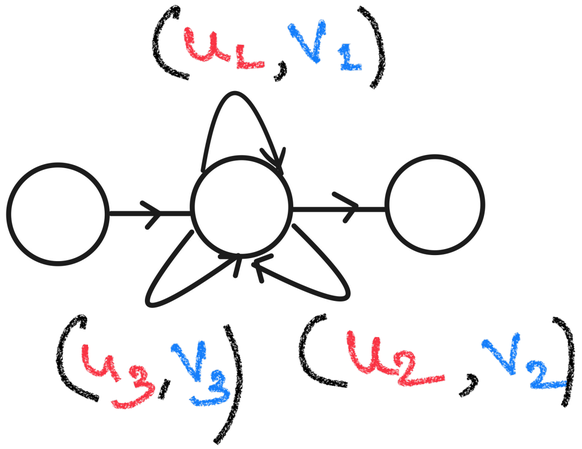


$(abca, baac)$

More on conjugates



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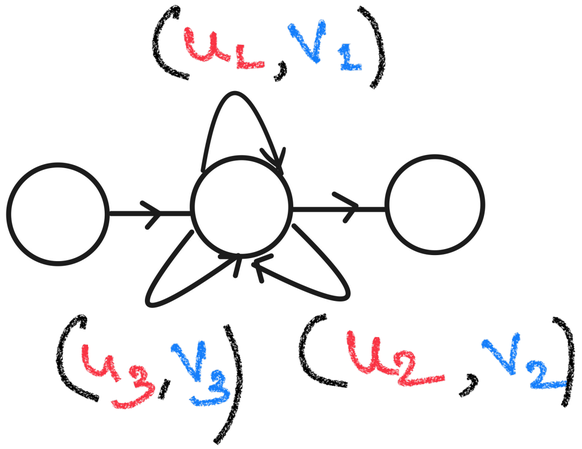

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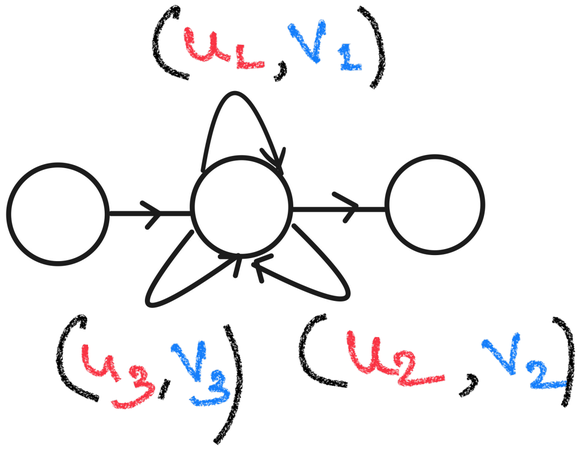
More on conjugates



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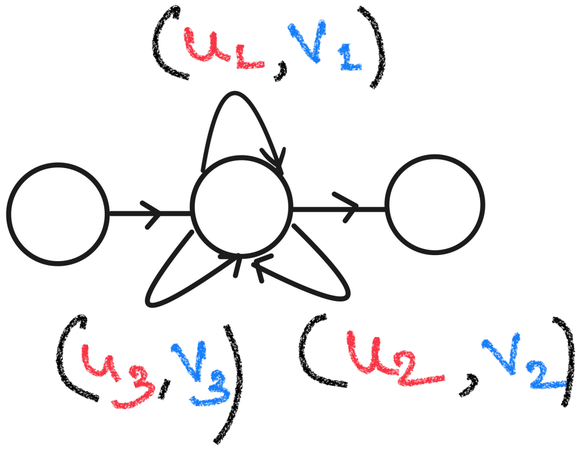
More on conjugates



where each u_i and v_i are conjugates

- $G = \{(u_i, v_i) \mid i \in I\}$
- Is $\langle G \rangle$ conjugate?
- $\langle G \rangle = \{(u_{i_1} u_{i_2} \cdots u_{i_n}, v_{i_1} v_{i_2} \cdots v_{i_n}) \mid n \geq 1, i_j \in I\}$

Conjugates

- Theorem (Lyndon Schutzenberger)

$uz = zv$ **iff** $u = xy$ **and** $v = yx$ **and** $z \in (xy)^*x$

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Conjugates

- Theorem (Lyndon Schutzenberger)

$$uz = zv \quad \textbf{iff} \quad u = xy \textbf{ and } v = yx \textbf{ and } z \in (xy)^*x$$
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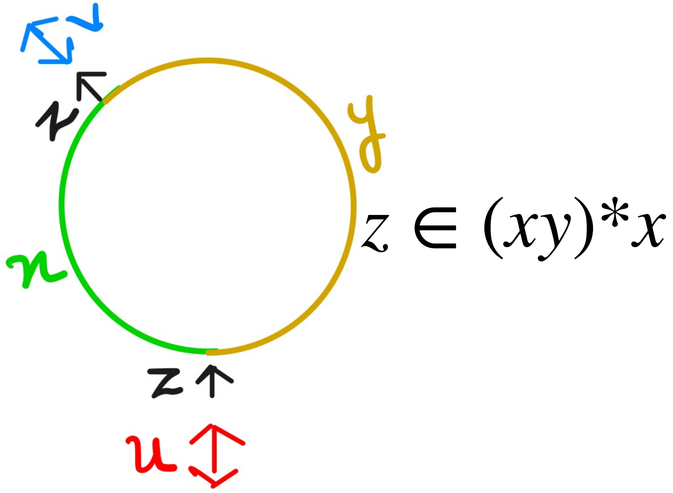
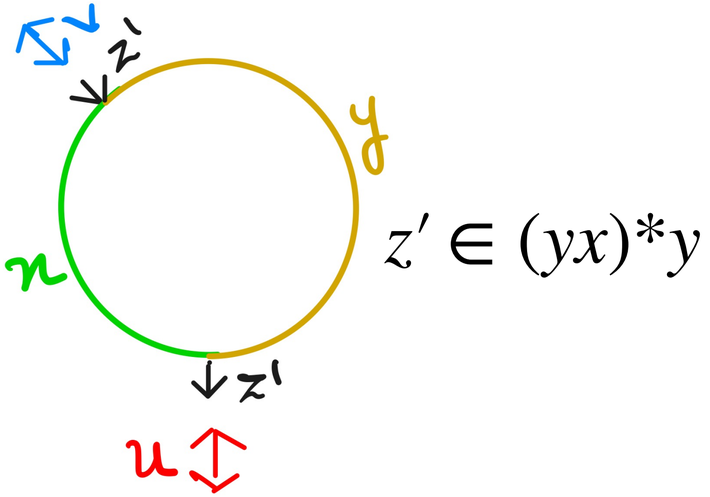


$z \in (xy)^*x$

z is an inner-witness of $(u\ ,\ v\ )$



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Conjugates

- Common Witness Theorem

$\langle G \rangle$ **is conjugate iff** $\exists z : \; z \in \cap \, (x_i y_i)^* x_i$

Conjugates

- Common Witness Theorem

$\langle G \rangle$ **is conjugate iff** $\exists z : \; z \in \cap\,(x_i y_i)^* x_i$ **OR** $z \in \cap\,(y_i x_i)^* y_i$

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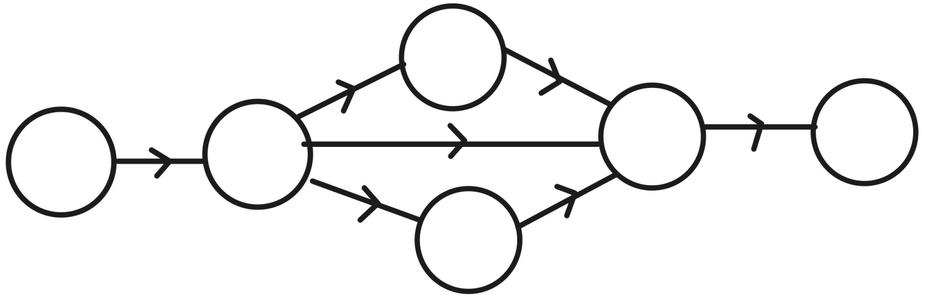

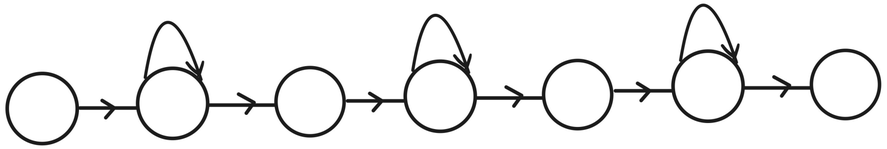

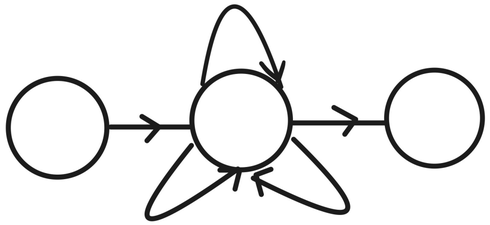

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For Levenstein

- Get a rational expression from the cartesian product
- Express it as a sum of sum-free expressions
- Structural induction. For star, do a Check conjugacy of $\langle G \rangle$.

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For Levenshtein

- Two transducers are close w.r.t. Levenshtein edit distance iff all the cycles in their Cartesian product are conjugates.

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- Two transducers are close w.r.t. Levenshtein edit distance iff all the cycles in their Cartesian product are conjugates.

Closeness w.r.t. Levenshtein edit distance reduces to deciding conjugacy of rational relations.

Closeness of transducers

- Let d be a metric on words. Lift it to transducers.

Distance is computable iff closeness and k-closeness are decidable. (for integer-valued metrics)

- Given T_1, T_2 is $d(T_1, T_2)$ computable? (Distance)
- Given T_1, T_2 is $d(T_1, T_2)$ finite? (Closeness)
- Given T_1, T_2 and $k \in \mathbb{N}$, is $d(T_1, T_2)$ at most k? (k-closeness)

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} **Decidable**

Thank you